Chapter 5

Nonlinear Systems

This chapter considers problems that involve two first-order ordinary differential equations, at least one of which is nonlinear. These problems are usually difficult enough that finding a formula for the solution is not possible. Consequently, most of the chapter does not concern solving these problems, but instead concentrates on developing ways to determine the properties of the solution. What this means will be explained as the methods are derived. We begin with examples that illustrate the problems we will be considering.

Example 1: Pendulum

The equation for the angular deflection of a pendulum is (see Figure 5.1)

$$\ell\frac{d^2\theta}{dt^2} = -g\sin\theta, \tag{5.1}$$

where the initial angle $\theta(0)$ and the initial angular velocity $\theta'(0)$ are assumed to be given. Also, ℓ is the length of the pendulum and g is the gravitational acceleration constant. Introducing the angular velocity $v = \theta'$ then the equation can be written as the first-order system

$$\theta' = v, \tag{5.2}$$

$$v' = -\alpha\sin\theta, \tag{5.3}$$

where $\alpha = g/\ell$. Although (5.2) is linear, (5.3) is nonlinear because of the term $\sin\theta$. Consequently, together (5.2), (5.3) form a nonlinear first-order system for θ and v. ■

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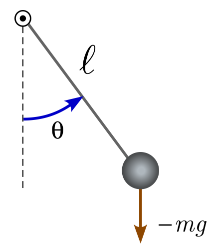

Figure 5.1. *Angular deflection of a pendulum.*

Example 2: Measles

A model for the spread of a disease, like measles, is

$$\frac{dS}{dt} = \alpha N - (\beta I + \alpha)S,$$
$$\frac{dI}{dt} = \beta IS - (\alpha + \gamma)I.$$

In these equations, $S(t)$ is the number of people susceptible to the disease, and $I(t)$ is number that are ill. The nonlinearity, which is due to the term IS, appears in both equations. ■

Nonlinear systems are usually not solvable using elementary functions. What is possible it to ask questions about the solution that are significant and answerable. As an example, with measles, a reasonable question would be: what would it take to eliminate the disease from the population? This requires that $I \to 0$ as $t \to \infty$ (and the faster this happens the better). In more mathematical terms, we want $I = 0$ to be an asymptotically stable steady state. How to modify the stability of $I = 0$, with the goal of quickly eliminating the disease, will be considered in Section 5.2.2.

A question arising with the pendulum is, does it ever stop moving? Given the physical assumptions used in the derivation of the equation it is reasonable to expect that it does not stop and, in fact, the solution is expected to be periodic. So, we would like to know if it is possible to show that the solution is periodic, and in the process determine the period without actually solving the problem.

5.1 ▪ Non-Linear Systems

The problems in this chapter can be written in component form as

$$u' = f(u,v), \tag{5.4}$$
$$v' = g(u,v). \tag{5.5}$$

In these equations, $u(t)$ and $v(t)$ are the dependent variables, and f and g are given functions of u and v. It is assumed that the equations are *autonomous*, which means that f and g do not depend explicitly on t.

The vector form of (5.4), (5.5) is

$$\frac{d\mathbf{y}}{dt} = \mathbf{f}(\mathbf{y}), \tag{5.6}$$

where

$$\mathbf{y} = \begin{pmatrix} u \\ v \end{pmatrix}, \quad \text{and} \quad \mathbf{f} = \begin{pmatrix} f(u,v) \\ g(u,v) \end{pmatrix}.$$

For an initial value problem, an initial condition of form

$$\mathbf{y}(0) = \begin{pmatrix} u_0 \\ v_0 \end{pmatrix}. \tag{5.7}$$

would also be given.

Example: For the nonlinear system

$$u' = v - \frac{1}{2}u, \tag{5.8}$$

$$v' = -\frac{1}{2}v + 2u(2-u^2), \tag{5.9}$$

we have that

$$\mathbf{f} = \begin{pmatrix} v - \frac{1}{2}u \\ -\frac{1}{2}v + 2u(2-u^2) \end{pmatrix}.$$

There are no known simple mathematical methods that can be used to find the solution of this system (by hand). However, it is easily solved using a computer, and four example curves are shown in Figure 5.2. In all four cases, the solution ends up at one of two points. In this chapter we will not attempt to find the solution curves but we will be very interested in determining these two points and finding the reason why the solution approaches them. ■

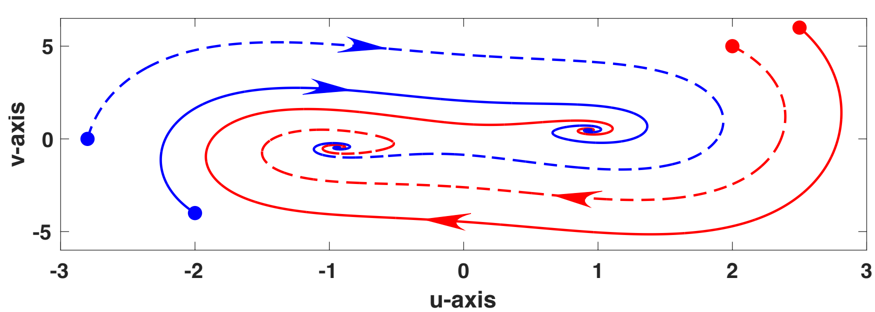

Figure 5.2. *Solution curves of (5.8), (5.9) in the u,v-plane for four different initial conditions (shown with the solid dots). The arrows indicate the direction for increasing t. [video]*

5.1.1 ▪ Steady-State Solutions

For $\mathbf{y}' = \mathbf{f}(\mathbf{y})$, a **steady state solution** $\mathbf{y}_s$ is a constant vector that satisfies $\mathbf{f}(\mathbf{y}_s) = \mathbf{0}$. In component form, the requirements are that

$$f(u_s, v_s) = 0, \tag{5.10}$$

$$g(u_s, v_s) = 0. \tag{5.11}$$

Solving for u_s and v_s is not straightforward. In fact, given that $f(u,v)$ and $g(u,v)$ can be almost anything, there is no method that always works for solving these equations. The recommendation is to pick one of the equations, and use it to solve for u in terms of v, or v in terms of u. The equation to pick for this is usually the one that is easiest to solve. This solution is then substituted into the other equation, and you then have one equation and one unknown (see Example 1). It is also not uncommon that you need to be opportunistic, and take advantage of certain terms in the equation to help simply the equations (see Example 2).

Example 1: Find the steady states of

$$\frac{du}{dt} = 3 - u - v - uv,$$
$$\frac{dv}{dt} = uv - 2v.$$

Answer: The equations to solve are

$$3 - u - v - uv = 0,$$
$$uv - 2v = 0.$$

The second equation looks the easiest to work with. Factoring it as $v(u-2) = 0$, we get two solutions: $v = 0$ and $u = 2$. Taking $v = 0$, then from the first equation we get that $u = 3$. For $u = 2$, from the first equation we get that $v = 1/3$. Therefore, we have found two steady states: $(u_s, v_s) = (3, 0)$, and $(u_s, v_s) = (2, 1/3)$. ■

Example 2: Assuming α is a positive constant, find the steady states of

$$\frac{du}{dt} = 1 - (1+\alpha)u + u^2 v,$$
$$\frac{dv}{dt} = u - u^2 v.$$

Answer: The equations to solve are

$$1 - (1+\alpha)u + u^2 v = 0,$$
$$u - u^2 v = 0.$$

It is possible to use the approach from the previous example, but it is easier to look a little closer at these equations. They both contain the term u^2v. In fact, from the second equation $u^2v = u$. Using this information in the first equation, we get that $u = 1/\alpha$. From the second equation, it follows that $v = \alpha$. Therefore, we have found that the only steady state is: $(u_s, v_s) = (1/\alpha, \alpha)$. ■

Example 3: Find the steady states of

$$\begin{aligned} x' &= x - x^2 - xy, \\ y' &= 2y - y^2 - 3xy. \end{aligned}$$

Answer: The equations to solve are

$$\begin{aligned} x - x^2 - xy &= 0, \\ 2y - y^2 - 3xy &= 0. \end{aligned}$$

Factoring the first equation as $x(1-x-y) = 0$, then either $x = 0$ or $x = 1 - y$. If $x = 0$, then from the second equation $y = 0$ or $y = 2$, giving us the two steady states $(0, 0)$ and $(0, 2)$. When $x = 1 - y$, the second equation reduces to $y(1 - 2y) = 0$, which has solutions $y = 0$ and $y = 1/2$. This gives us two more steady states, which are $(1/2, 1/2)$ and $(1, 0)$. ■

Example 4: For the system

$$\begin{aligned} x' &= x - y, \\ y' &= (x - y)^3, \end{aligned}$$

the steady states are any points that satisfy $y = x$. ■

Example 5: The system

$$\begin{aligned} x' &= x + y, \\ y' &= 1, \end{aligned}$$

does not have a steady state solution because it is not possible for $y' = 0$. ■

We are going to avoid the situation in Example 4. Specifically, in the problems we will consider, there can be multiple steady states, but they are discrete points as in Examples 1, 2, and 3. The way this will be stated is that the steady states are isolated, which means that there is a nonzero distance d so that the distance between any two steady states for the problem is at least d.

Exercises

1. Write the following as $\mathbf{y}' = \mathbf{f}(\mathbf{y})$, making sure to identify the entries in $\mathbf{y}$ and $\mathbf{f}$. If initial conditions are given, write them as $\mathbf{y}(0) = \mathbf{y}_0$.

a) $u' = u^2 - v$
$v' = 2u - 3v$

b) $u' = u^2 + v^2$
$2v' = \sin(u)$

c) $u' = e^u - v$
$v' = uv$
$u(0) = -1,\ v(0) = 0$

d) Van der Pol oscillator
$u'' + (1 - u^2)u' + u = 0$

e) Toda oscillator
$u'' + e^u - 1 = 0$

f) Duffing oscillator
$u'' + u + u^3 = 0$
$u(0) = 1,\ u'(0) = -1$

g) Michaelis-Menten system
$S' = -k_1ES + k_{-1}(E_0 - E),$
$E' = -k_1ES + (k_2 + k_{-1})(E_0 - E)$
$S(0) = 1,\ E(0) = 2$

h) Predator-prey
$x' = ax - bxy$
$y' = -cy + dxy$

i) Projectile (nonuniform field)
$$y'' = -\frac{gR^2}{(R+y)^2}$$
$y(0) = 0,\ y'(0) = 3$

j) Orbital motion
$$r'' = \frac{\alpha^2}{r^3} - \frac{\mu}{r^2}$$
$r(0) = 1,\ r'(0) = 2$

2. Find the steady state solutions of the following.

a) $\begin{cases} u' = 1 - 2u - v - uv \\ v' = 3uv - v \end{cases}$

b) $\begin{cases} u' = v - u^2 \\ v' = v + u^3 \end{cases}$

c) $\begin{cases} u' = 4 - uv^2 \\ v' = -v + uv^2 \end{cases}$

d) $\begin{cases} S' = 2S - S^2 - \frac{2SP}{1+S} \\ P' = \frac{2SP}{1+S} - P \end{cases}$

e) $\begin{cases} S' = -IS + 5 - I - S \\ I' = IS - I \end{cases}$

f) $\begin{cases} s' = c - s^2 \\ c' = 1 + sc \end{cases}$

g) $\begin{cases} x' = \sin(y) + \sin(x) \\ y' = 3y^2 + x^4 \end{cases}$

h) $\begin{cases} x' = xy \\ y' = (2 - x - y)(1 + y) \end{cases}$

i) $\begin{cases} x' = x(4 - x - y) \\ y' = y(6 - y - 3x) \end{cases}$

j) $\begin{cases} u' = v + u(u^2 + v^2) \\ v' = -u + v(u^2 + v^2) \end{cases}$

5.2 ▪ Stability

The question considered now is central to this chapter, and it is whether a steady state is achievable. What this means is that the steady state is asymptotically stable. To explain how we are going to determine stability,